Linear Transformations Which Leave Controllable Multiinput Descriptor Systems Controllable

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ABSTRACT

A general linear descriptor system $E\dot{x} = Ax + Bu$, where $E, A \in \mathbb{C}^{n,n}$, $B \in \mathbb{C}^{n,m}$, $x \in \mathbb{C}^n$, $u \in \mathbb{C}^m$, E singular, is called controllable if $\operatorname{rank}[\alpha E - \beta A, B] = n$ for all $(\alpha, \beta) \neq (0,0)$, $\alpha, \beta \in \mathbb{C}$. Let $f: \mathbb{C}^{n,n+m} \to \mathbb{C}^{n,n+m}$ be a linear transformation of the form $f(X) = CXD$. We characterize all such linear transformations that leave the set $\mathscr{C} = \{[\alpha E - \beta A, B] | \operatorname{rank}[\alpha E - \beta A, B] = n \ \forall (\alpha, \beta) \neq (0,0),\ \alpha, \beta \in \mathbb{C}\}$ invariant and show that only the well-known transformations that leave controllability invariant can occur.

1. INTRODUCTION

We study general autonomous linear systems of the form

$$E\dot{x}(t) = Ax(t) + Bu(t), \qquad y = Cx, \tag{1.1}$$

where $E, A \in \mathbb{C}^{n,n}$, $B \in \mathbb{C}^{n,m}$, $C \in \mathbb{C}^{p,n}$ are constant matrices, $u(t) \in \mathbb{C}^m$, $x(t) \in \mathbb{C}^n$, $y(t) \in \mathbb{C}^p$ are time dependent vectors, $n, m, p \in \mathbb{N}$ and E is possibly singular.

If E is singular, such systems are called descriptor systems or singular systems. Here $u(t)$ is a control function, and $x(t)$ is called the state of the system at time t. Solvability, controllability, observability, and other system theoretic properties of such systems have been extensively studied in recent

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years; see for example [1, 7, 8, 11]. In this paper we will mainly discuss linear transformations of the system (1.1) that leave these system theoretic properties invariant. In the following we will mainly discuss systems where $\alpha E - \beta A$ is a regular pencil, i.e., $\det(\alpha E - \beta A) \not\equiv 0$ for $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$. Such systems are often called *solvable* systems. Pairs $(\alpha, \beta) \in \mathbb{C}^2$ where $\det(\alpha E - \beta A) = 0$ are called *eigenvalues*, and we identify all the pairs $t(\alpha, \beta)$ $\forall t \in \mathbb{C} \setminus \{0\}$. A pair $(\alpha, 0)$ is an infinite eigenvalue. We use this pair notation for eigenvalues to include the infinite eigenvalues. Conditions when a nonsolvable system can be transformed into a solvable system by a constant feedback $u = Fx$ are given in [1].

A system (1.1) is called

controllable if $\operatorname{rank}[\alpha E - \beta A, B] = n$ $\forall(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$;
observable if $\operatorname{rank}[(\alpha E - \beta A)^*, C^*] = n$ $\forall(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$.

For singular systems also other types of controllability are discussed in the literature, but we will only examine the type given above. In order to simplify the analysis of such systems, one often transforms the system into canonical forms or other simplified versions, using linear transformations, that leave the structural properties of the system invariant. For the standard case

$$\dot{x} = Ax + Bu, \qquad y = Cx, \tag{1.2}$$

i.e. systems with E equal to the identity matrix I, one has the following well-known result (e.g. Wonham [10] or Wimmer [9]).

THEOREM 1.3.

(a) *Let the system* (1.2) *be controllable. Then for all* $R \in \mathbb{C}^{n,n}$, $V \in C^{m,m}$, R, V *nonsingular, and for all* $W \in \mathbb{C}^{m,n}$ *the system*

$$\dot{z} = (RAR^{-1} + RBWR^{-1})z + RBV^{-1}v, \qquad y = CR^{-1}z \tag{1.4}$$

is controllable, where $z = Rx$, $v = Vu$.

(b) *Let the system* (1.2) *be observable. Then for all* $R \in \mathbb{C}^{n,n}$, $V \in C^{p,p}$, R, V *nonsingular and for all* $W \in \mathbb{C}^{n,p}$, *the system*

$$\dot{z} = (RAR^{-1} + RWCR^{-1})z + RBu, \qquad w = VCR^{-1}z \tag{1.5}$$

is observable, where $z = Rx$, $w = Vy$.

It is very easy to extend this result to the general case, and we will do so in the next section.

Having Theorem 1.3, an obvious question is: Are there other transformations that one can perform with the system (1.1) such that controllability or observability or both are preserved? We do not know the general answer to this question, but we will show that for a certain type of linear transformations, the transformations described in Theorem 1.3 are the only ones, even for the general case.

2. EQUIVALENT CONDITIONS FOR CONTROLLABILITY

In the standard case (1.2) several equivalent conditions are known for controllability of a system, e.g. [5, 6]. In order to prove results on controllability it is very useful to be able to switch between different conditions. We therefore extend these results to the general case.

THEOREM 2.1. *Let $\alpha E - \beta A \in \mathbf{C}^{n,n}$ be a regular pencil and $B \in \mathbf{C}^{n,m}$. Then the following are equivalent:*

(i) *The system* (1.1) *is controllable.*

(ii) *If $P, Q \in \mathbf{C}^{n,n}$ are such that $P(\alpha E - \beta A)Q$ is in Kronecker canonical form* (*e.g.* [2]),

$$\alpha\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} - \beta\begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \begin{matrix} \}\, n_1 \\ \}\, n_2 \end{matrix}, \tag{2.2}$$

and

$$PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

is partitioned analogously, then

$$\operatorname{rank}\left[B_1, JB_1, \ldots, J^{n_1-1}B_1\right] = n_1$$

and

$$\operatorname{rank}\left[B_2, NB_2, \ldots, N^{n_2-1}B_2\right] = n_2.$$

(iii) *There do not exist nonsingular $P, Q \in \mathbf{C}^{n,n}$ such that*

$$P(\alpha E - \beta A)Q = \begin{bmatrix} \alpha E_{11} - \beta A_{11} & \alpha E_{12} - \beta A_{12} \\ 0 & \alpha E_{22} - \beta A_{22} \end{bmatrix} \tag{2.3}$$

and

$$PB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \tag{2.4}$$

where the partitioning is analogous.

(iv) *There does not exist a left eigenvector $y \neq 0$ of $\alpha E - \beta A$ satisfying $y^* B = 0$.*

(v) *There exist $P, Q \in \mathbf{C}^{n,n}$ nonsingular such that $P(\alpha E - \beta A)Q$ is of the form*

(2.5)

$$\alpha \begin{bmatrix} I & & & & & \\ & \ddots & & & & \\ & & I & & & \\ & & & N_1 & & \\ & & & & \ddots & \\ & & & & & N_k \end{bmatrix} - \beta \begin{bmatrix} J_1 & & & & & \\ & \ddots & & & & \\ & & J_l & & & \\ & & & I & & \\ & & & & \ddots & \\ & & & & & I \end{bmatrix},$$

where the blocks J_i are Jordan blocks and the blocks N_i are nilpotent Jordan blocks. If PB is partitioned accordingly as

$$PB = \begin{bmatrix} B_{J_1} \\ \vdots \\ B_{J_l} \\ B_{N_1} \\ \vdots \\ B_{N_k} \end{bmatrix}, \tag{2.6}$$

then the blocks of PB have the following property: If p Jordan blocks

$J_{i_1},\ldots,J_{i_p}$ have the same eigenvalue, then the rank of the matrix

$$C=\left[c_{i_1},\ldots,c_{i_p}\right] \tag{2.7}$$

is p, where the columns of C are the conjugate transposes of the last rows of $B_{J_{i_1}},\ldots,B_{J_{i_p}}$. If nilpotent blocks $N_1,\ldots,N_k$ exist, then the matrix

$$D=\left[d_1,\ldots,d_k\right] \tag{2.8}$$

has rank k, where d_j is the conjugate transpose of the last row of B_{N_j}.

Proof. Parts of this proof are given in [11], and other parts can be obtained from the proof in the standard case $E=I$. We therefore keep the proof very brief.

(i) $\Leftrightarrow$ (ii): See the proof in [11].

(i) $\Rightarrow$ (iii): Suppose there exist $P,Q\in\mathbf{C}^{n,n}$, nonsingular, such that

$$P(\alpha E-\beta A)Q=\begin{bmatrix}\alpha E_{11}-\beta A_{11} & \alpha E_{12}-\beta A_{12}\\ 0 & \alpha E_{22}-\beta A_{22}\end{bmatrix}$$

and

$$PB=\begin{bmatrix}B_1\\ 0\end{bmatrix}$$

with analogous partitioning. Then the matrix

$$W=\begin{bmatrix}\alpha E_{11}-\beta A_{11} & B_1 & \alpha E_{12}-\beta A_{12}\\ 0 & 0 & \alpha E_{22}-\beta A_{22}\end{bmatrix}$$

has rank less than n for each $(\alpha,\beta)\in\mathbf{C}^2\setminus\{(0,0)\}$ eigenvalue of $\alpha E-\beta A$, which contradicts (i).

(iii) $\Rightarrow$ (iv): Suppose there exists $y\neq 0$ such that $y^*(\alpha_0E-\beta_0A)=0$ for $(\alpha_0,\beta_0)\in\mathbf{C}^2\setminus\{(0,0)\}$ and $y^*B=0$. Let P,Q be nonsingular such that PAQ, PEQ are upper triangular, with eigenvalue (α_0,β_0) in the bottom diagonal position. Such P,Q always exist by the generalized Schur theorem (e.g. Golub and Van Loan [3]). Let $\tilde{y}^* = y^*P^{-1}$, $\tilde{B} = PB$; then $\tilde{y}^*\tilde{B} = 0$. But $\tilde{y} = e_n$, where e_n denotes the nth unit vector. Thus

$$\tilde{B} = \begin{bmatrix} & B_1 & \\ 0 & \cdots & 0 \end{bmatrix},$$

which contradicts (iii).

(iv) $\Rightarrow$ (i): Trivial.

(v) $\Leftrightarrow$ (i): Let P, Q be nonsingular, such that $P(\alpha E - \beta A)Q$ is in Kronecker canonical form (e.g. [2])

$$(2.9)\quad \alpha \begin{bmatrix} I & & & & & \\ & \ddots & & & & \\ & & I & & & \\ & & & N_1 & & \\ & & & & \ddots & \\ & & & & & N_k \end{bmatrix} - \beta \begin{bmatrix} J_1 & & & & & \\ & \ddots & & & & \\ & & J_l & & & \\ & & & I & & \\ & & & & \ddots & \\ & & & & & I \end{bmatrix},$$

and let PB be partitioned accordingly as

$$PB = \begin{bmatrix} B_{J_1} \\ \vdots \\ B_{J_l} \\ B_{N_1} \\ \vdots \\ B_{N_k} \end{bmatrix}.$$

If there exist p blocks $J_{j_1}, \ldots, J_{j_p}$ with the same eigenvalue (α_0, β_0), (i.e., the quotient α_0/β_0 is equal for all these blocks), then $\operatorname{rank}[P(\alpha_0 E - \beta_0 A)Q, PB] = n$ implies that the submatrix corresponding to the last rows of $J_{j_1}, \ldots, J_{j_p}$ has full rank, but in $P(\alpha E - \beta A)Q$ it is a zero matrix. Thus, the corresponding rows of PB have to form a matrix of full rank. The same argument clearly holds for several nilpotent blocks. Conversely, if the corresponding rows in PB always have full rank, then the rank of $[P(\alpha E - \beta A)Q, PB]$ is full. ■

It should be noted that in the single input case ($m = 1$), there are also further equivalent conditions. For the sake of completeness we state the corresponding result in this case here, too.

THEOREM 2.10. *Let $\alpha E - \beta A \in \mathbb{C}^{n,n}$ be a regular pencil and let $b \in \mathbb{C}^n$. Then the following are equivalent:*

(i) *The system* (1.1) *is controllable.*
(ii) *If $P, Q \in \mathbb{C}^{n,n}$ are such that*

$$P(\alpha E - \beta A)Q = \alpha\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} - \beta\begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}\begin{matrix} \}\, n_1 \\ \}\, n_2 \end{matrix} \tag{2.11}$$

and

$$Pb = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

is partitioned analogously, then

$$\operatorname{rank}[b_1, Jb_1, \dots, J^{n_1-1}b_1] = n_1 \quad \textit{and} \quad \operatorname{rank}[b_2, Nb_2, \dots, N^{n_2-1}b_2] = n_2.$$

(iii) *There do not exist nonsingular $P, Q \in \mathbb{C}^{n,n}$ such that*

$$P(\alpha E - \beta A)Q = \begin{bmatrix} \alpha E_{11} - \beta A_{11} & \alpha E_{12} - \beta A_{12} \\ 0 & \alpha E_{22} - \beta A_{22} \end{bmatrix} \tag{2.12a}$$

and

$$Pb = \begin{bmatrix} b_1 \\ 0 \end{bmatrix}, \tag{2.12b}$$

where the partitioning is analogous.

(iv) *There does not exist a left eigenvector $y \neq 0$ of $\alpha E - \beta A$ satisfying $y^*b = 0$.*

(v) *There exist P, Q nonsingular such that*

$$P(\alpha E - \beta A)Q = \alpha\begin{bmatrix} I & & & \\ & \ddots & & \\ & & I & \\ & & & N \end{bmatrix} - \beta\begin{bmatrix} J_1 & & & \\ & \ddots & & \\ & & J_l & \\ & & & I \end{bmatrix}, \tag{2.13}$$

where J_i, $i = 1, \dots, l$ are Jordan blocks to pairwise different eigenvalues

(α_i, β_i), $\beta_i \neq 0$ *(i.e.* $\alpha_i/\beta_i \neq \alpha_j/\beta_j$ *if* $i \neq j$*), and* N *is one single nilpotent Jordan block. Furthermore*

$$Pb = \begin{bmatrix} b_{J_1} \\ \vdots \\ b_{J_l} \\ b_N \end{bmatrix}, \qquad \textit{where} \quad b_N = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad b_{J_i} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \; i = 1, \ldots, l.$$

P and Q are unique except for a block permutation of the blocks J_i. (This form is called descriptor Lur'e-Lefschetz canonical form).

(vi) *For any $\tilde{E}$, $\tilde{A}$, $\tilde{b}$ of the same dimensions as E, A, b, also satisfying*

$$\text{(2.14)} \qquad \operatorname{rank}\left[\alpha\tilde{E} - \beta\tilde{A}, \tilde{b}\right] = n \qquad \textit{for all } (\alpha, \beta) \neq (0,0)$$

and

$$\text{(2.15)} \qquad \det[\alpha E - \beta A] = \det\left[\alpha\tilde{E} - \beta\tilde{A}\right],$$

there exist nonsingular $P, Q \in \mathbb{C}^{n,n}$ such that

$$\text{(2.16)} \qquad P(\alpha E - \beta A)Q = \alpha\tilde{E} - \beta\tilde{A}, \qquad Pb = \tilde{b}.$$

(vii) *There exist unique $P, Q \in \mathbb{C}^{n,n}$ nonsingular such that*

$$PEQ = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \qquad PAQ = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}, \qquad Pb = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

with the same partitioning in all three terms, where N is one single nilpotent Jordan block,

$$C = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ \gamma_1 & \gamma_2 & \cdots & \gamma_l \end{bmatrix},$$

and b_1, b_2 *are of the form*

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Proof. The proof can be obtained almost analogously to the proof in the case $E = I$ (e.g. [5, p. 319]) or follows immediately from the previous theorem.

Analogous results can be obtained for observability of systems, and we omit them here.

3. MAIN RESULTS

We now prove the main results of this paper, beginning with the extension of Theorem 1.3 to the singular case.

THEOREM 3.1. *Let* $E, A \in \mathbf{C}^{n,n}$, $B \in \mathbf{C}^{n,m}$, $\alpha E - \beta A$ *a regular pencil, and let the system*

$$E\dot{x} = Ax + Bu \tag{3.2}$$

be controllable. Then for any $P, Q \in \mathbf{C}^{n,n}$, $W \in \mathbf{C}^{m,m}$ *nonsingular and for any* $F, G \in \mathbf{C}^{m,n}$ *the systems*

$$PEQ\dot{z} = PAQz + PBu \qquad (z = Q^{-1}x), \tag{3.3}$$

$$E\dot{x} = Ax + BWv \qquad (v = W^{-1}u), \tag{3.4}$$

$$E\dot{x} = (A + BF)x + Bv \qquad (v = u - Fx), \tag{3.5}$$

$$(E + BG)\dot{x} = Ax + Bv \qquad (v = u - G\dot{x}) \tag{3.6}$$

are also controllable.

Proof. For all $(\alpha, \beta) \in \mathbf{C}^2 \setminus \{(0,0)\}$ the following ranks are equal:

$$\begin{aligned}\operatorname{rank}[\alpha E - \beta A, B] &= \operatorname{rank}\left(P[\alpha E - \beta A, B]\begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix}\right)\\ &= \operatorname{rank}\left([\alpha E - \beta A, B]\begin{bmatrix} I & 0 \\ 0 & W \end{bmatrix}\right)\\ &= \operatorname{rank}\left([\alpha E - \beta A, B]\begin{bmatrix} I & 0 \\ -\beta F & I \end{bmatrix}\right)\\ &= \operatorname{rank}\left([\alpha E - \beta A, B]\begin{bmatrix} I & 0 \\ \alpha G & I \end{bmatrix}\right).\end{aligned}$$

So the systems are controllable. ∎

Note that for a controllable and solvable system

$$E\dot{x} = Ax + Bu$$

a feedback can be such that

$$\operatorname{rank}[\alpha E - \beta A, B] = n \qquad \text{for all} \quad (\alpha, \beta) \in \mathbf{C}^2 \setminus \{(0,0)\}$$

but $\det(\alpha E - \beta A) \equiv 0$, as the following example shows. Consider the system

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u,$$

which is clearly controllable by Theorem 2.1, but for $u = [0, -1]x + v$ we obtain the system

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}v.$$

Obviously $\det(\alpha E - \beta A) \equiv 0$ for this system, while

$$\operatorname{rank}\left[\alpha\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \beta\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right] = 2$$

for all $(\alpha, \beta) \neq (0,0)$.

Thus, for general systems one has to be careful with the transformations given in (3.5), (3.6), and for the standard system with $E = I$ one has to be careful with the transformation given in (3.6). The latter is almost never considered in the standard case, because it may, as mentioned above, turn a "nice" linear system into a "nasty" descriptor system. But there may be applications where this can turn out useful. For general systems these transformations can be used to make (for example) unsolvable systems solvable and also achieve other structural properties. See for example [1] or [4].

Analogous observations and a result analogous to Theorem 3.1 can clearly be obtained also for observable systems.

We will now show that for a certain class of linear transformations, the transformations in (3.3)–(3.6) are the only ones that leave all controllable systems controllable. The result will follow as a corollary from the result for the standard case ($E = I$), which we will prove first.

THEOREM 3.7. *Let*

$$f: \mathbb{C}^{n,n+m} \to \mathbb{C}^{n,n+m}$$
$$: X \mapsto UXV,$$

where $U \in \mathbb{C}^{n,n}$, $V \in \mathbb{C}^{n+m,n+m}$, *and* $m \leqslant n$. *Assume that for any controllable system*

$$\dot{x} = Ax + Bu \tag{3.8}$$

the transformed system

$$\dot{x} = \tilde{A}x + \tilde{B}u \tag{3.9}$$

is also controllable, where

$$[\tilde{A}, \tilde{B}] = U[A, B]V. \tag{3.10}$$

Then U is nonsingular and

$$V = \begin{bmatrix} tU^{-1} & 0 \\ 0 & I \end{bmatrix} \tilde{V}, \tag{3.11}$$

where $\tilde{V}$ is a product of matrices of the following types:

$$\begin{bmatrix} I & 0 \\ F & I \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & W \end{bmatrix} \tag{3.12}$$

with $W \in \mathbb{C}^{m,m}$ nonsingular and $F \in \mathbb{C}^{m,n}$ arbitrary.

Proof. Partition V as

$$\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix},$$

where $V_{11} \in \mathbb{C}^{n,n}$, $V_{22} \in \mathbb{C}^{m,m}$, $V_{12}, V_{21}^T \in \mathbb{C}^{n,m}$. Let

$$V_{22} = W_1^* \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_m \end{bmatrix} W_2^*, \qquad \sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_m, \tag{3.13}$$

be the singular value decomposition of V_{22} with W_1, W_2 unitary (e.g. [3]), and let Q be nonsingular such that

$$\tilde{V}_{21} = W_1 V_{21} Q^{-1} = \begin{bmatrix} 0 & \cdots & 0 & * & \cdots & * \\ & & & & \ddots & \vdots \\ & & & & & * \end{bmatrix} \tag{3.14}$$

(Q can for example be obtained from an RQ decomposition of W_1V_{21}; see e.g. [3]). It is immediate that U is invertible, since otherwise the rank would be decreased, and we now show that also V_{22} is invertible. Suppose $\tilde{V}_{22} = W_1V_{22}W_2$ is singular; then $\sigma_m = 0$. Let

$$\begin{aligned} A &= U^{-1}Q^{-1} \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} QU, \\ B &= U^{-1}Q^{-1} \begin{bmatrix} & & & 0 \\ & 0 & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} W_1, \end{aligned} \tag{3.15}$$

which define a controllable system by Theorem 2.1. Thus it follows that

$$\tilde{A} = QUAV_{11}Q^{-1} + QUBW_1^{-1}W_1V_{21}Q^{-1} \tag{3.16}$$

has a last row of the form $[0 \ \cdots \ 0 \ x]$, and

$$\tilde{B} = QUAV_{12}W_2 + QUBW_1^{-1}W_1V_{22}W_2 \tag{3.17}$$

has a zero last row. Thus, for $(\alpha, \beta) = (x, 1)$ we have $\operatorname{rank}[\alpha I - \beta\tilde{A}, \tilde{B}] < n$, i.e., $[\tilde{A}, \tilde{B}]$ is not controllable. Thus, it follows that V_{22} is invertible.

Suppose now that $V_{12} \neq 0$. Then let $Q \in \mathbb{C}^{n,n}$, $W_2 \in \mathbb{C}^{m,m}$ unitary such that

$$QUV_{12}V_{22}^{-1}W_2 = \begin{bmatrix} 0 & & \\ \vdots & & \\ 0 & & \\ \sigma_m & & \\ & \ddots & \\ & & \sigma_1 \end{bmatrix} \tag{3.18}$$

defines a singular value decomposition with $\sigma_1 \neq 0$. Let

$$A = U^{-1}Q^{-1}\begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}QU, \qquad B = -AV_{12}V_{22}^{-1}. \tag{3.19}$$

Then

(3.20)

$$[A, B] = U^{-1}Q^{-1}\left[\begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}, \begin{bmatrix} & 0 & \\ \sigma_m & & \\ & \ddots & \\ & & \sigma_1 \end{bmatrix}\right]\begin{bmatrix} QU & 0 \\ 0 & W_2^{-1} \end{bmatrix}$$

defines a controllable system, since $\sigma_1 \neq 0$. But

$$U[A, B]V = U[A, B]\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$$

$$(3.21) \qquad = Q^{-1}\left[\begin{bmatrix} 1 & 1 & & & \\ & \ddots & \ddots & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} QU(V_{11} - V_{12}V_{21}^{-1}V_{21}), 0\right],$$

which clearly cannot belong to a controllable system. Thus it follows that $V_{12} = 0$.

By Theorem 3.1 we may now transform the system w.l.o.g. to the controllable system defined by

$$(3.22) \qquad U^{-1}U[A, B]V\begin{bmatrix} U & 0 \\ -V_{22}^{-1}V_{21} & V_{22}^{-1} \end{bmatrix} = [AV_{11}U, B].$$

Let

$$(3.23) \qquad J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_l \end{bmatrix} = QV_{11}UQ^{-1}$$

be the Jordan canonical form of $V_{11}U$. If J contains Jordan blocks to different eigenvalues, say w.l.o.g. J_1, J_2, then let

$$J = \begin{bmatrix} J_1 & & \\ & J_2 & \\ & & \hat{J} \end{bmatrix}, \qquad \text{where } \hat{J} = \begin{bmatrix} J_3 & & \\ & \ddots & \\ & & J_l \end{bmatrix},$$

and let

$$(3.24) \qquad A = Q^{-1}\begin{bmatrix} \tilde{J}_2 & & \\ & \tilde{J}_1 & \\ & & \tilde{J} \end{bmatrix}Q$$

with the same partitioning as J, where $\tilde{J}_2$ is a Jordan block having the same eigenvalue as J_2, $\tilde{J}_1$ is a Jordan block with the same eigenvalue as J_1, and $\tilde{J}$ is one single Jordan block with an eigenvalue different to those of J_1, J_2. Let

$$B = Q^{-1}\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}, \tag{3.25}$$

again with the same partitioning as above, where the B_i's are blocks with a 1 in the lower right corner and zero elsewhere. By Theorem 2.1 $[A, B]$ defines a controllable system. Then

$$\begin{aligned}[\tilde{A}, \tilde{B}] &= Q[AV_{11}U, B]\begin{bmatrix} Q^{-1} & 0 \\ 0 & I \end{bmatrix} \\ &= \left[\begin{bmatrix} \tilde{J}_1 J_2 & & \\ & \tilde{J}_2 J_1 & \\ & & J\tilde{J} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}\right].\end{aligned} \tag{3.26}$$

If $(\alpha, \beta) = (\lambda, 1)$, where λ is the eigenvalue of $\tilde{J}_1 J_2$, then

$$\operatorname{rank}\left[\alpha I - \beta\tilde{A}, \tilde{B}\right] < n, \tag{3.27}$$

i.e., the transformed system is not controllable. Thus $V_{11}U$ has only Jordan blocks to the same eigenvalue. Suppose Jordan blocks of size greater than 1 to the eigenvalue λ of $V_{11}U$ exist, then let

$$\begin{aligned} A &= Q^{-1}\begin{bmatrix} -\lambda & 1 & & \\ & \ddots & \ddots & \\ & & -\lambda & 1 \\ & & & -\lambda \end{bmatrix}Q, \\ B &= Q^{-1}\begin{bmatrix} & & & 0 \\ & 0 & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}, \end{aligned} \tag{3.28}$$

such that $\tilde{J} = QV_{11}UQ^{-1}$ is in Jordan canonical form. Again $[A, B]$ defines a controllable system. The matrix

$$\begin{bmatrix} -\lambda & 1 & & & \\ & \ddots & \ddots & & \\ & & -\lambda & 1 & \\ & & & -\lambda \end{bmatrix} \tilde{J} \tag{3.29}$$

has at least two Jordan blocks to the eigenvalue $-\lambda^2$, but the rank of the matrix formed from the corresponding rows in $\tilde{B}$ is only 1, which by Theorem 2.1 is a contradiction. Thus $V_{11}U = tI$, which finishes the proof. ■

Note that for the case $m > n$, by Theorem 3.1 we can find a nonsingular $Q \in \mathbb{C}^{m,m}$ such that

$$BQ = \begin{bmatrix} * & & & & & \\ \vdots & \ddots & & & & \\ * & \cdots & * & 0 & \cdots & 0 \end{bmatrix} =: [B_1, 0],$$

where $B_1 \in \mathbb{C}^{n,n}$. Defining a new control function $\tilde{u}(t) \in \mathbb{C}^n$ by the first n elements of $Q^{-1}u(t)$, we can reduce the system to one with $m \leqslant n$. Thus the assumption in Theorem 3.7 can be relaxed to arbitrary m.

The result for the general case now follows as a corollary.

COROLLARY 3.30. *Let*

$$f: \mathbb{C}^{n,n+m} \to \mathbb{C}^{n,n+m}$$
$$: X \mapsto UXV,$$

where $U \in \mathbb{C}^{n,n}$, $V \in \mathbb{C}^{n+m,n+m}$, *such that* $\operatorname{rank}(U[\alpha E - \beta A, B]V) = n$ *for all* $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$, *and for all controllable systems*

$$E\dot{x} = Ax + Bu. \tag{3.31}$$

Then U *is nonsingular and*

$$V = \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \tilde{V}, \tag{3.32}$$

where $Q \in \mathbb{C}^{n,n}$ *is nonsingular and* $\tilde{V}$ *is a product of matrices of the types*

$$\begin{bmatrix} I & 0 \\ F & I \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & W \end{bmatrix}, \tag{3.33}$$

where $F \in \mathbb{C}^{n,m}$ *is arbitrary and* $W \in \mathbb{C}^{m,m}$ *nonsingular.*

Proof. Consider all controllable systems (3.31) with E nonsingular. Then

$$\operatorname{rank}(U[\alpha E - \beta A, B]V) = n \qquad \text{for all} \quad (\alpha, \beta) \notin \mathbb{C}^2 \setminus \{(0,0)\} \tag{3.34}$$

if and only if

$$\operatorname{rank}\left(U\left[\frac{\alpha}{\beta}I - AE^{-1}, B\right]\begin{bmatrix} \beta E & 0 \\ 0 & I \end{bmatrix}V\right) = n \tag{3.35}$$

$$\text{for all} \quad (\alpha, \beta) \in \mathbb{C}^2, \quad \beta \neq 0.$$

By Theorem 3.7 it follows that

$$\begin{bmatrix} \beta E & 0 \\ 0 & I \end{bmatrix}V$$

has to be a product of matrices

$$\begin{bmatrix} tU^{-1} & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} I & 0 \\ F & I \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & W \end{bmatrix};$$

hence V has to be a product of matrices

$$\begin{bmatrix} \beta t E^{-1}U^{-1} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} I & 0 \\ F & I \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & W \end{bmatrix}$$

for arbitrary nonsingular Q, since E can be chosen arbitrary nonsingular. ■

Note that it is obvious how to produce an analogous result for observability.

4. CONCLUSION

We have characterized a special class of linear transformations that leave the controllability of general linear systems invariant. We do not know whether this result holds for more general linear transformations

$$f: \mathbb{C}^{n,n+m} \to \mathbb{C}^{n,n+m},$$

but we conjecture that Theorem 3.7 and Corollary 3.30 also hold in the general case.

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